

ON THE CRAW–ISHII CONJECTURE

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ABSTRACT. In [2], Craw and Ishii proved that for a finite abelian group $G \subset \mathrm{SL}_3(\mathbb{C})$ every (projective) relative minimal model of \mathbb{C}^3/G is isomorphic to the fine moduli space \mathcal{M}_θ of θ -stable G -constellations for some GIT parameter θ . In this article, we conjecture that the same is true for a finite group $G \subset \mathrm{GL}_3(\mathbb{C})$ if a relative minimal model Y of $X = \mathbb{C}^3/G$ is smooth. We prove this for some abelian groups.

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1. INTRODUCTION

Let G be a finite subgroup in $\mathrm{GL}_n(\mathbb{C})$. A G -cluster Z is a G -invariant subscheme of \mathbb{C}^n with $H^0(\mathcal{O}_Z)$ isomorphic to $\mathbb{C}[G]$ the regular representation of G . For a finite group G in $\mathrm{SL}_2(\mathbb{C})$, Ito and Nakamura [8] showed that the minimal resolution of \mathbb{C}^2/G is isomorphic to the G -Hilbert scheme $G\text{-Hilb } \mathbb{C}^2$ that is the fine moduli space of G -clusters.

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In [1], Bridgeland, King and Reid proved that for a finite group G in $\mathrm{SL}_3(\mathbb{C})$ the G -Hilbert scheme $G\text{-Hilb } \mathbb{C}^3$ is a crepant resolution of \mathbb{C}^3/G .

In [2], Craw and Ishii introduced a generalised notion of G -clusters. A G -constellation \mathcal{F} is a G -equivariant sheaf on \mathbb{C}^n with $H^0(\mathcal{F})$ isomorphic to $\mathbb{C}[G]$. Define the GIT stability parameter space

$$\Theta = \{ \theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0 \},$$

where $R(G)$ denotes the representation space of G . For $\theta \in \Theta$, we say that G -constellation \mathcal{F} is θ -(semi)stable if $\theta(\mathcal{G}) > 0$ ($\theta(\mathcal{G}) \geq 0$) for every nonzero proper subsheaf \mathcal{G} of \mathcal{F} . A stability parameter θ is called *generic* if every θ -semistable G -constellation is θ -stable.

Furthermore, Craw and Ishii constructed the moduli space \mathcal{M}_{θ} of θ -stable G -constellations using GIT [2]. They conjectured that for a finite subgroup $G \subset \mathrm{SL}_3(\mathbb{C})$, every projective crepant resolution of \mathbb{C}^3/G is isomorphic to \mathcal{M}_{θ} for some $\theta \in \Theta$ and proved this for G being abelian. Note that if $G \subset \mathrm{SL}_3(\mathbb{C})$, then \mathbb{C}^3/G has Gorenstein canonical singularities. Being motivated by this, this article makes the following conjecture and proves the conjecture for some cases.

Conjecture 1.1 (Craw–Ishii conjecture). *Let G be a finite subgroup in $\mathrm{GL}_3(\mathbb{C})$. Suppose $X = \mathbb{C}^3/G$ has a smooth relative minimal model. Then every relative minimal model of X is isomorphic to (an irreducible component of) \mathcal{M}_{θ} for a suitable GIT parameter θ .*

On the other hand, \mathcal{M}_{θ} need not be irreducible in general [5]. However, if G is abelian, Craw, Maclagan and Thomas [4] showed that \mathcal{M}_{θ} has a unique irreducible component Y_{θ} containing the torus $(\mathbb{C}^{\times})^n/G$ for generic θ . Furthermore, they proved that Y_{θ} can be obtained by variation of GIT from \mathbb{C}^n/G . The component Y_{θ} is called the *birational component* of \mathcal{M}_{θ} .

Theorem 1.2 (Main Theorem). *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{r}(1, a, b)$ with b coprime to a satisfying one of the following:*

- (i) $r = abc + a + b + 1$ for some positive integer c ;
- (ii) $r = abc + a - 2b + 1$ and $b = ak + 1$ for some positive integers c, k with $c \geq 2$ and $a \geq 3$.

Then every relative minimal model $Y \rightarrow X := \mathbb{C}^3/G$ is isomorphic to the birational component Y_{θ} of the moduli space \mathcal{M}_{θ} of θ -stable G -constellations for a suitable parameter θ .

Moreover the main theorem implies that the relative minimal model can be obtained by variation of GIT from \mathbb{C}^3/G for the cases.

Corollary 1.3. *In the situation as in Theorem 1.2, every relative minimal model $Y \rightarrow X = \mathbb{C}^3/G$ is obtained by variation of GIT quotient.*

1.1. Overview of the article. Let G be a finite group in $\mathrm{GL}_3(\mathbb{C})$. We say that $\varphi: Y \rightarrow X := \mathbb{C}^3/G$ is a *relative minimal model* if:

- (i) Y has only \mathbb{Q} -factorial terminal singularities;
- (ii) K_Y is φ -nef;
- (iii) φ is projective.

For example, for the case where $G \subset \mathrm{SL}_3(\mathbb{C})$, a projective crepant resolution $Y \rightarrow \mathbb{C}^3/G$ is a relative minimal model.

For the group G of type $\frac{1}{r}(\alpha_1, \dots, \alpha_n)$, i.e.

$$G = \langle \mathrm{diag}(\epsilon^{\alpha_1}, \dots, \epsilon^{\alpha_n}) \mid \epsilon^r = 1 \rangle \subset \mathrm{GL}_n(\mathbb{C}),$$

by toric geometry, the quotient variety $X = \mathbb{C}^n/G$ is given by the toric cone

$$\sigma_+ := \mathrm{Cone}(e_1, \dots, e_n)$$

with the lattice

$$L = \mathbb{Z}^n + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \dots, \alpha_n).$$

Fix a primitive interior lattice point $v = \frac{1}{r}(a_1, \dots, a_n) \in L \cap \sigma_+$. The *star subdivision* of σ_+ at v is the minimal fan containing the following n -dimensional cones σ_k for $k = 1, \dots, n$:

$$\sigma_k := \mathrm{Cone}(e_1, \dots, \hat{e}_k, v, \dots, e_n).$$

Then the corresponding toric variety X_v admits the induced projective toric morphism $\nu: X_v \rightarrow X = \mathbb{C}^n/G$. If v generates L/\mathbb{Z}^n , then the affine open set U_k of X_v corresponding to σ_k has a quotient singularity \mathbb{C}^n/G_k for some abelian group G_k , eg. G_1 is the group of type $\frac{1}{a_1}(-r, a_2, \dots, a_n)$. Note that the order of G_k is smaller than the order of G . Thus we can use induction on the order of groups.

The groups in the main theorem satisfy:

- (i) every relative minimal model $Y \rightarrow X$ is smooth;
- (ii) every relative minimal model $Y \rightarrow X$ has a projective morphism $Y \rightarrow X_v$ for $v = \frac{1}{r}(1, a, b)$.

For the proof, the notion of G -bricks and round down functions is essential, which was recently developed in [9, 10].

A G -brick Γ is a certain \mathbb{C} -basis of $H^0(\mathcal{F})$ for a torus invariant G -constellation \mathcal{F} on the birational component (see Definition 2.7). We say that Γ is θ -stable if the corresponding G -constellation \mathcal{F} is θ -stable. Using suitable G -bricks, we are able to describe an affine local chart of the birational component Y_θ (see Theorem 2.13).

The *round down functions for the star subdivisions at v* are maps between monomial lattices compatible with the star subdivision. Using the round down functions, we produce a set \mathfrak{S} of G -bricks from G_k -bricks.

Since the set \mathfrak{S} is a G -brickset (see Definition 2.15), it suffices to find a GIT parameter θ such that every G -brick $\Gamma \in \mathfrak{S}$ is θ -stable.

After finding a parameter θ , we conclude that Y is isomorphic to Y_θ for some θ .

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2. G -CONSTELLATIONS AND G -BRICKS

2.1. Moduli spaces of G -constellations. In this section, we briefly review moduli spaces of θ -stable G -constellations (see e.g. [2, 4, 11]).

Consider a finite diagonal group G in $\mathrm{GL}_n(\mathbb{C})$.

Definition 2.1. A G -equivariant coherent sheaf \mathcal{F} on \mathbb{C}^n is called a G -constellation if $H^0(\mathcal{F})$ is isomorphic to the regular representation $\mathbb{C}[G]$ of G as a $\mathbb{C}[G]$ -module.

Remark 2.2. For a free G -orbit Z in \mathbb{C}^3 , \mathcal{O}_Z is a G -constellation. \diamond

Define the GIT stability parameter space

$$\Theta = \{ \theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0 \}$$

where $R(G) := \bigoplus_{\rho \in \mathrm{Irr} G} \mathbb{Z} \cdot \rho$ is the representation space of G .

Definition 2.3. For a stability parameter $\theta \in \Theta$, we say that:

- (i) a G -constellation \mathcal{F} is θ -semistable if $\theta(\mathcal{G}) \geq 0$ for every subsheaf $\mathcal{G} \subsetneq \mathcal{F}$;
- (ii) a G -constellation \mathcal{F} is θ -stable if $\theta(\mathcal{G}) > 0$ for every subsheaf $0 \neq \mathcal{G} \subsetneq \mathcal{F}$;
- (iii) θ is generic if every θ -semistable object is θ -stable.

By King [11], it is known that if θ is generic, then there exists a quasiprojective scheme \mathcal{M}_θ which is a fine moduli space of θ -stable G -constellations.

Moreover, in [7], Ito–Nakajima showed that \mathcal{M}_θ is canonically isomorphic to $G\text{-Hilb } \mathbb{C}^n$ for $\theta \in \Theta_+$ where

$$(2.4) \quad \Theta_+ := \{ \theta \in \Theta \mid \theta(\rho) > 0 \text{ for } \rho \neq \rho_0 \}.$$

In particular, \mathcal{M}_θ can be obtained by variation of GIT from $G\text{-Hilb } \mathbb{C}^n$.

Assume that θ is generic. Let \mathcal{M}_θ denote the fine moduli space of θ -stable G -constellations. Craw, MacLagan and Thomas showed that the moduli space \mathcal{M}_θ need not be irreducible [5]. Furthermore, they proved that \mathcal{M}_θ has a distinguished component Y_θ which is birational to \mathbb{C}^n/G if G is abelian [4].

Theorem 2.5 (Craw–Maclagan–Thomas [4]). *Assume that G be a finite abelian group in $\mathrm{GL}_n(\mathbb{C})$. For a generic parameter $\theta \in \Theta$, the moduli space \mathcal{M}_θ has a unique irreducible component Y_θ that contains the torus $T := (\mathbb{C}^\times)^n/G$. Moreover:*

- (i) Y_θ is a not-necessarily-normal toric variety which is birational to the quotient variety \mathbb{C}^n/G ;
- (ii) there is a projective morphism $Y_\theta \rightarrow \mathbb{C}^n/G$ obtained by variation of GIT quotient.

Definition 2.6. The unique irreducible component Y_θ in Theorem 2.5 is called the *birational component* of \mathcal{M}_θ .

2.2. Cyclic quotients and toric lattices. Consider the group G of type $\frac{1}{r}(\alpha_1, \dots, \alpha_n)$, i.e.

$$G = \langle \mathrm{diag}(\epsilon^{\alpha_1}, \dots, \epsilon^{\alpha_n}) \mid \epsilon^r = 1 \rangle \subset \mathrm{GL}_n(\mathbb{C}).$$

As G is abelian, the set of irreducible representations of G can be identified with the character group $G^\vee := \mathrm{Hom}(G, \mathbb{C}^\times)$ of G .

For the group G of type $\frac{1}{r}(\alpha_1, \dots, \alpha_n)$, define the lattice

$$L = \mathbb{Z}^n + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \dots, \alpha_n).$$

Set $\bar{L} = \mathbb{Z}^n \subset L$. Consider the two dual lattices $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, $\bar{M} = \mathrm{Hom}_{\mathbb{Z}}(\bar{L}, \mathbb{Z})$. Note that we can consider the two dual lattices \bar{M} and M as Laurent monomials and G -invariant Laurent monomials, respectively.

The embedding of G into the torus $(\mathbb{C}^\times)^n \subset \mathrm{GL}_n(\mathbb{C})$ induces a surjective homomorphism

$$\mathrm{wt}: \bar{M} \longrightarrow G^\vee$$

with kernel M . Note that there are two isomorphisms of abelian groups $L/\mathbb{Z}^n \rightarrow G$ and $\bar{M}/M \rightarrow G^\vee$.

Let $\bar{M}_{\geq 0}$ denote genuine monomials in \bar{M} , i.e.

$$\bar{M}_{\geq 0} = \{x_1^{m_1} \cdots x_n^{m_n} \in \bar{M} \mid m_i \geq 0 \text{ for all } i\}.$$

For a set $A \subset \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$, $\langle A \rangle$ denotes the $\mathbb{C}[x_1, \dots, x_n]$ -submodule of $\mathbb{C}[x_1^\pm, \dots, x_n^\pm]$ generated by A .

Let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbb{Z}^n and σ_+ the cone generated by e_1, \dots, e_n . By toric geometry, the corresponding affine toric variety $U_{\sigma_+} = \mathrm{Spec} \mathbb{C}[\sigma_+^\vee \cap M]$ is the quotient variety $X = \mathbb{C}^3/G$.

2.3. G -bricks and the birational component Y_θ . In this section, we review the notion of G -bricks introduced in [9, 10]. Using G -bricks, we can describe an affine local chart of the birational component Y_θ .

Definition 2.7. A G -prebrick Γ is a subset of Laurent monomials in $\mathbb{C}[x_1^\pm, \dots, x_n^\pm]$ satisfying:

- (i) the monomial $\mathbf{1}$ is in Γ ;

- (ii) for each weight $\rho \in G^\vee$, there exists a unique Laurent monomial $\mathbf{m}_\rho \in \Gamma$ of weight ρ , i.e. $\text{wt}: \Gamma \rightarrow G^\vee$ is bijective;
- (iii) if $\mathbf{p}' \cdot \mathbf{p} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{p}, \mathbf{p}' \in \overline{M}_{\geq 0}$, then $\mathbf{p} \cdot \mathbf{m}_\rho \in \Gamma$;
- (iv) the set Γ is *connected* in the sense that for any element \mathbf{m}_ρ , there is a (fractional) path in Γ from \mathbf{m}_ρ to $\mathbf{1}$ whose steps consist of multiplying or dividing by one of x_i .

For a Laurent monomial $\mathbf{m} \in \overline{M}$, let $\text{wt}_\Gamma(\mathbf{m})$ denote the unique element \mathbf{m}_ρ in Γ of the same weight as \mathbf{m} .

For a G -prebrick $\Gamma = \{\mathbf{m}_\rho\}$, we define $S(\Gamma)$ to be the subsemigroup of M generated by $\frac{\mathbf{p} \cdot \mathbf{m}_\rho}{\text{wt}_\Gamma(\mathbf{p} \cdot \mathbf{m}_\rho)}$ for all $\mathbf{p} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. We define a cone $\sigma(\Gamma)$ in $L_\mathbb{R} = \mathbb{R}^n$ as follows:

$$\begin{aligned} \sigma(\Gamma) &= S(\Gamma)^\vee \\ &= \left\{ \mathbf{u} \in L_\mathbb{R} \mid \left\langle \mathbf{u}, \frac{\mathbf{p} \cdot \mathbf{m}_\rho}{\text{wt}_\Gamma(\mathbf{p} \cdot \mathbf{m}_\rho)} \right\rangle \geq 0, \quad \forall \mathbf{m}_\rho \in \Gamma, \mathbf{p} \in \overline{M}_{\geq 0} \right\}. \end{aligned}$$

As is proved in [10], the semigroup $S(\Gamma)$ is finitely generated as a semigroup. Thus the semigroup $S(\Gamma)$ defines an affine toric variety. Define two affine toric varieties:

$$\begin{aligned} U(\Gamma) &:= \text{Spec } \mathbb{C}[S(\Gamma)], \\ U^\vee(\Gamma) &:= \text{Spec } \mathbb{C}[\sigma(\Gamma)^\vee \cap M]. \end{aligned}$$

Definition 2.8. For a G -prebrick Γ ,

$$B(\Gamma) := \{x_i \cdot \mathbf{m}_\rho \mid \mathbf{m}_\rho \in \Gamma, \} \setminus \Gamma$$

is called the *Border bases* of Γ .

Let Γ be a G -prebrick. Define

$$C(\Gamma) := \langle \Gamma \rangle / \langle B(\Gamma) \rangle.$$

The module $C(\Gamma)$ is a torus invariant G -constellation. A submodule \mathcal{G} of $C(\Gamma)$ is determined by a subset $A \subset \Gamma$, which forms a \mathbb{C} -basis of \mathcal{G} .

Lemma 2.9. *Let A be a subset of Γ . The following are equivalent.*

- (i) *The set A forms a \mathbb{C} -basis of a submodule of $C(\Gamma)$.*
- (ii) *If $\mathbf{m}_\rho \in A$, then $x_i \cdot \mathbf{m}_\rho \in \Gamma$ implies $x_i \cdot \mathbf{m}_\rho \in A$ for all i .*

Definition 2.10. Let Γ be a G -prebrick.

- (i) A G -prebrick Γ is called a G -brick if the affine toric variety $U(\Gamma)$ contains a torus fixed point.
- (ii) A G -prebrick Γ is called θ -stable if the torus invariant G -constellation $C(\Gamma)$ is θ -stable.

Note that from toric geometry, $U(\Gamma)$ has a torus fixed point if and only if $S(\Gamma) \cap (S(\Gamma))^{-1} = \{\mathbf{1}\}$, i.e. the cone $\sigma(\Gamma)$ is an n -dimensional cone.

Proposition 2.11 ([9]). *For generic θ , let Γ be a θ -stable G -brick and Y_θ the birational component of \mathcal{M}_θ . There exists an open immersion*

$$U(\Gamma) = \operatorname{Spec} \mathbb{C}[S(\Gamma)] \hookrightarrow Y_\theta.$$

Remark 2.12. The G -brick Γ forms a \mathbb{C} -basis of G -constellations parametrised by $U(\Gamma)$. \diamond

Theorem 2.13 ([9]). *Let $G \subset \operatorname{GL}_n(\mathbb{C})$ be a finite diagonal group and θ a generic GIT parameter for G -constellations. Assume that \mathfrak{S} is the set of all θ -stable G -bricks.*

- (i) *The birational component Y_θ of \mathcal{M}_θ is isomorphic to the not-necessarily-normal toric variety $\bigcup_{\Gamma \in \mathfrak{S}} U(\Gamma)$.*
- (ii) *The normalisation of Y_θ is isomorphic to the normal toric variety whose toric fan consists of the n -dimensional cones $\sigma(\Gamma)$ for $\Gamma \in \mathfrak{S}$ and their faces.*

Remark 2.14. For given G and θ , it is difficult to find all θ -stable G -bricks in general. \diamond

Definition 2.15. For a finite diagonal group $G \subset \operatorname{GL}_n(\mathbb{C})$, assume that Y is a normal toric variety admitting a proper birational morphism $Y \rightarrow X := \mathbb{C}^n/G$. Let Σ_{\max} denote the set of the n -dimensional cones in the fan of Y . A set \mathfrak{S} of G -bricks is called a G -brickset for Y if \mathfrak{S} satisfies:

- (i) there is a bijective map $\Sigma_{\max} \rightarrow \mathfrak{S}$ sending σ to Γ_σ ;
- (ii) $S(\Gamma_\sigma) = \sigma^\vee \cap M$.

Proposition 2.16. *Suppose that $Y \rightarrow X := \mathbb{C}^n/G$ is a proper birational morphism. Let \mathfrak{S} be a G -brickset for Y . Then Y is isomorphic to the toric variety $\bigcup_{\Gamma \in \mathfrak{S}} U(\Gamma)$. Moreover, if there exists $\theta \in \Theta$ such that every Γ in \mathfrak{S} is θ -stable, then Y is isomorphic to Y_θ .*

Proof. By definition, it is clear that Y is isomorphic to the toric variety $\bigcup_{\Gamma \in \mathfrak{S}} U(\Gamma)$. Assume that there exists $\theta \in \Theta$ such that every Γ in \mathfrak{S} is θ -stable. From Proposition 2.11, we can conclude that there exists an open immersion $\iota: Y \hookrightarrow Y_\theta$. Furthermore, since $Y \rightarrow X$ is proper and $Y_\theta \rightarrow X$ is projective, ι is a closed embedding between n -dimensional toric varieties. Thus ι is an isomorphism. \square

3. STAR SUBDIVISIONS AND MODULI DESCRIPTIONS

3.1. Star subdivisions and round down functions. Fix a primitive lattice point $v = \frac{1}{r}(a_1, \dots, a_n) \in L \cap \sigma_+$. The *star subdivision* (or *barycentric subdivision*) Σ of σ_+ at v is the minimal fan containing all cones $\operatorname{Cone}(\tau, v)$ where τ varies over all faces of σ_+ with $v \notin \tau$. Let $X := U_{\sigma_+}$ be the affine toric variety corresponding to σ_+ and X_Σ the toric variety corresponding to the fan Σ . Then the star subdivision

induces a projective toric morphism $\nu: X_\Sigma \rightarrow X = \mathbb{C}^n/G$ with the ramification formula

$$(3.1) \quad rK_{X_\Sigma} - \nu^*rK_X \equiv_{\text{num}} \left(\sum_i a_i - r \right) E_v,$$

where E_v is the torus invariant prime divisor corresponding to the 1-dimensional cone $\text{Cone}(v)$.

The fan Σ consists of the n -dimensional cone σ_k and its faces for $k = 1, \dots, n$:

$$\sigma_k := \text{Cone}(e_1, \dots, \hat{e}_k, v, \dots, e_n).$$

Assume that v generates L/\mathbb{Z}^n . Fix $k \in \{1, \dots, n\}$. Let L_k be

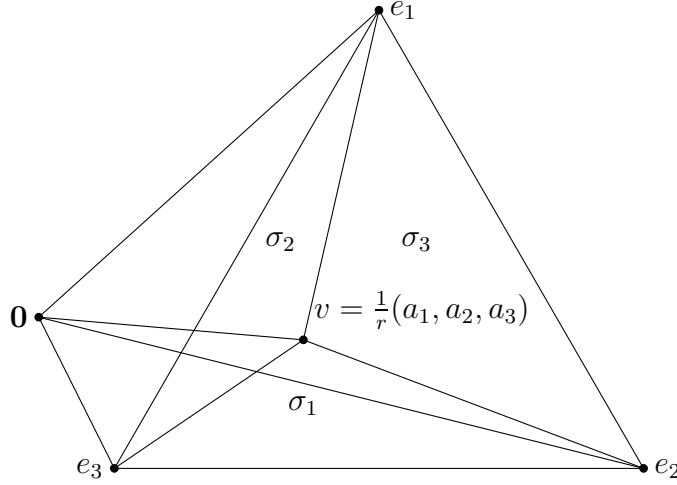


FIGURE 3.1. Star subdivision Σ of σ_+ at v

the sublattice of L generated by $e_1, \dots, \hat{e}_k, v, \dots, e_n$. Let us consider the dual lattice $M_k := \text{Hom}_{\mathbb{Z}}(L_k, \mathbb{Z})$ with the corresponding dual basis $\{\xi_1, \dots, \xi_n\}$

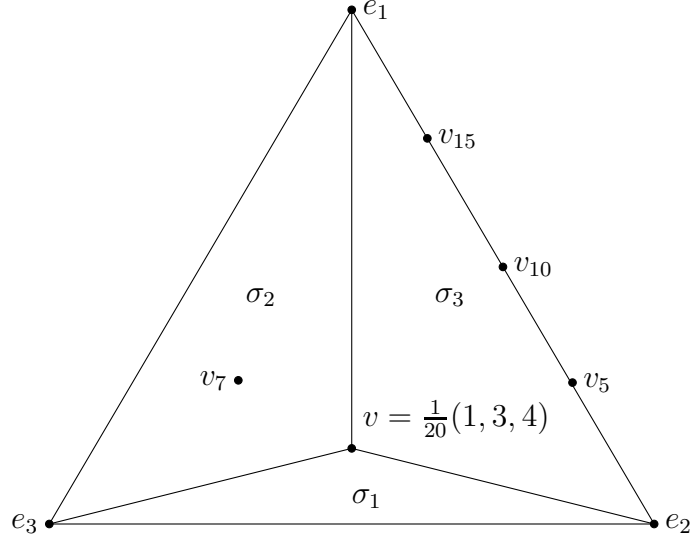
$$\xi_j = \begin{cases} x_j x_k^{-\frac{a_j}{a_k}} & \text{if } j \neq k, \\ x_k^{\frac{r}{a_k}} & \text{if } j = k. \end{cases}$$

Note that M_k contains the lattice M and that the lattice inclusion $L_k \hookrightarrow L$ induces a toric morphism

$$\varphi: \text{Spec } \mathbb{C}[\sigma_k^\vee \cap M_k] \rightarrow U_k := \text{Spec } \mathbb{C}[\sigma_k^\vee \cap M].$$

With eigencoordinates $\{\xi_1, \dots, \xi_n\}$, the toric affine variety U_k has a quotient singularity of type

$$\frac{1}{a_k}(a_1, \dots, \underbrace{-r}_{k\text{th}}, \dots, a_n).$$

FIGURE 3.2. Star subdivision at v for the type $\frac{1}{20}(1, 3, 4)$

Example 3.2. Consider the group G of type $\frac{1}{20}(1, 3, 4)$. Figure 3.2 shows the star subdivision at $v = \frac{1}{20}(1, 3, 4)$.

The cone σ_2 corresponds to the quotient singularity of type $\frac{1}{3}(1, 1, 1)$ with eigencoordinates $xy^{-\frac{1}{3}}, y^{\frac{20}{3}}, y^{-\frac{4}{3}}z$. There exists a unique lattice point $v_7 = \frac{1}{20}(7, 1, 8)$ on the plane containing e_1, v, e_3 . On the other hand, the cone σ_3 on the right side of v has a singularity of type $\frac{1}{4}(1, 3, 0)$. Note that there are three other lattice points v_5, v_{10}, v_{15} on the plane containing e_1, e_2, v . Lastly, the affine toric variety corresponding to the cone $\sigma_1 = \text{Cone}(e_2, e_3, v)$ is smooth as v, e_2, e_3 form a \mathbb{Z} -basis of L . \diamond

Definition 3.3 (Round down functions¹). With the notation above, for $k \in \{1, \dots, n\}$, the k -th round down function $\phi_k: \overline{M} \rightarrow M_k$ of the star subdivision at $\frac{1}{r}(a_1, \dots, a_n)$ is defined by

$$\phi_k(x_1^{m_1} \cdots x_n^{m_n}) = \xi_1^{m_1} \cdots \xi_k^{\lfloor \frac{1}{r} \sum a_i m_i \rfloor} \cdots \xi_n^{m_n}.$$

where $\lfloor \cdot \rfloor$ is the floor function.

Observe that since $v \in L$, $\frac{1}{r} \sum a_i m_i$ is an integer if and only if the monomial $x_1^{m_1} \cdots x_n^{m_n}$ is G -invariant.

For the star subdivision at $\frac{1}{r}(a_1, \dots, a_n)$, let G_k denote the abelian group L/L_k for $k \in \{1, \dots, n\}$.

Lemma 3.4. For each k , let ϕ_k be the round down function of the star subdivision at $\frac{1}{r}(a_1, \dots, a_n)$ and G_k the abelian group L/L_k . For a

¹As is stated in [9, 10], Davis, Logvinenko, and Reid [6] introduced a similar construction in a more general setting.

G -invariant monomial $\mathbf{p} \in M$ and a monomial $\mathbf{m} \in \overline{M}$,

$$\phi_k(\mathbf{m} \cdot \mathbf{p}) = \phi_k(\mathbf{m}) \cdot \mathbf{p}.$$

In particular, the weights of $\phi_k(\mathbf{m} \cdot \mathbf{p})$ and $\phi_k(\mathbf{m})$ are the same with respect to the G_k -action. Thus ϕ_k induces a well-defined surjective map

$$\phi_k: G^\vee \rightarrow G_k^\vee, \quad \rho \mapsto \phi_k(\rho),$$

where $\phi_k(\rho)$ is the weight of $\phi_k(\mathbf{m})$ for a monomial $\mathbf{m} \in \overline{M}$ of weight ρ .

Remark 3.5. In the lemma above, $\phi_k: G^\vee \rightarrow G_k^\vee$ can be described as follows. Let ρ_i be the irreducible representation of G whose weight is i . Then the weight of the representation $\phi_k(\rho_i)$ is j where j is the residue of i modulo a_k , i.e. $j = i \pmod{a_k}$. \diamond

Lemma 3.6. Let $\mathbf{m} \in \overline{M}$ be a monomial of weight j . The weight j satisfies $0 \leq j < r - a_k$ if and only if

$$\phi_k(x_k \cdot \mathbf{m}) = \phi_k(\mathbf{m}).$$

Proof. Assume that $\mathbf{m} = x_1^{m_1} \cdots x_n^{m_n}$ is a monomial of weight j with $0 \leq j < r - a_k$, i.e.

$$0 \leq \frac{1}{r} \sum a_i m_i - \lfloor \frac{1}{r} \sum a_i m_i \rfloor < \frac{r - a_k}{r}.$$

This is equivalent to the condition that $\phi_k(x_k \cdot \mathbf{m}) = \phi_k(\mathbf{m})$. \square

Lemma 3.7. If $\phi_k(\mathbf{m}) = \phi_k(\mathbf{m}')$ for some k , then $\mathbf{m} = \mathbf{p} \cdot \mathbf{m}'$ or $\mathbf{m}' = \mathbf{p} \cdot \mathbf{m}$ for some $\mathbf{p} \in \overline{M}_{\geq 0}$.

Proof. Let us suppose that $\phi_k(\mathbf{m}) = \phi_k(\mathbf{m}')$ for $\mathbf{m} = x_1^{m_1} \cdots x_n^{m_n}$ and $\mathbf{m}' = x_1^{m'_1} \cdots x_n^{m'_n}$ with $m_k \geq m'_k$. From the definition of the round down function ϕ_k , we have $m_i = m'_i$ for all $i \neq k$. Thus $\mathbf{m} = \mathbf{p} \cdot \mathbf{m}'$ with $\mathbf{p} = x_k^{m_k - m'_k} \in \overline{M}_{\geq 0}$. \square

Definition 3.8. The star subdivision of σ_+ at $v = \frac{1}{r}(a_1, \dots, a_n)$ is said to be *good* if:

- (i) v generates L/\mathbb{Z}^n ;
- (ii) for every $i \neq j$, $a_i + a_j \leq r$.

Lemma 3.9. Let ϕ_k be the k -th round down function of the good star subdivision at $\frac{1}{r}(a_1, \dots, a_n)$. For any monomial \mathbf{n} in the lattice M_k and any degree one monomial ξ_j in M_k , there exist x_i and a monomial $\mathbf{m} \in \overline{M}$ such that

$$\phi_k(x_i \cdot \mathbf{m}) = \xi_j \cdot \mathbf{n} \quad \text{with} \quad \phi_k(\mathbf{m}) = \mathbf{n}.$$

Proof. Fix k . Suppose that \mathbf{n} is a monomial in M_k and that ξ_j is a degree one monomial in M_k .

First consider the case where $j = k$, i.e. $\xi_j = \xi_k$. As the round down function ϕ_k is surjective, there exists $\mathbf{m} = x_1^{m_1} \cdots x_n^{m_n} \in \overline{M}$ such

that $\phi_k(\mathbf{m}) = \mathbf{n}$. By Lemma 3.6, after multiplying x_k enough, we may assume that $\phi_k(x_k \cdot \mathbf{m}) \neq \phi_k(\mathbf{m})$. This means that

$$\frac{1}{r} \sum_i a_i m_i + \frac{a_k}{r} \geq \lfloor \frac{1}{r} \sum_i a_i m_i \rfloor + 1.$$

Thus we have $\phi_k(x_k \cdot \mathbf{m}) = \xi_k \cdot \mathbf{n}$.

For the case where $j \neq k$, consider $\mathbf{m} = x_1^{m_1} \cdots x_n^{m_n} \in \overline{M}$ such that $\phi_k(\mathbf{m}) = \mathbf{n}$ with $\phi_k(x_k^{-1} \cdot \mathbf{m}) \neq \phi_k(\mathbf{m})$, i.e.

$$\frac{1}{r} \sum_i a_i m_i - \frac{a_k}{r} < \lfloor \frac{1}{r} \sum_i a_i m_i \rfloor.$$

Since the star subdivision is good, we have $a_k + a_j \leq r$. This implies that $\phi_k(x_j \cdot \mathbf{m}) = \xi_j \cdot \mathbf{n}$. \square

Proposition 3.10. *Let ϕ_k be the k -th round down function of the good star subdivision at $\frac{1}{r}(a_1, \dots, a_n)$. For a G_k -brick Γ' , define*

$$\Gamma := \{ \mathbf{m} \in \overline{M} \mid \phi_k(\mathbf{m}) \in \Gamma' \}.$$

- (i) *The set Γ is a G -brick with $S(\Gamma) = S(\Gamma')$.*
- (ii) *For $\mathbf{m} \in \overline{M}$, we have $\text{wt}_{\Gamma'}(\phi_k(\mathbf{m})) = \phi_k(\text{wt}_{\Gamma}(\mathbf{m}))$.*

Proof. First we show (ii) assuming that Γ is a G -prebrick. It follows that $\phi_k(\mathbf{m})$ is of the same weight as $\phi_k(\text{wt}_{\Gamma}(\mathbf{m}))$ from Lemma 3.4. Since $\phi_k(\text{wt}_{\Gamma}(\mathbf{m})) \in \Gamma'$, the assertion is proved.

To prove (i), note that $\mathbf{1} \in \Gamma$ as $\phi_k(\mathbf{1}) = \mathbf{1} \in \Gamma'$. Second we show that there exists a unique monomial of weight ρ in Γ for each $\rho \in G^\vee$. Fix $\rho \in G^\vee$. Since the star subdivision is good, we have a monomial $\mathbf{m} \in \overline{M}$ such that the weight of \mathbf{m} is ρ . Note that $\text{wt}_{\Gamma'}(\phi_k(\mathbf{m})) \in \Gamma'$ and that $\frac{\text{wt}_{\Gamma'}(\phi_k(\mathbf{m}))}{\phi_k(\mathbf{m})}$ is in the lattice M . From Lemma 3.4,

$$\phi_k: \mathbf{m} \cdot \left(\frac{\text{wt}_{\Gamma'}(\phi_k(\mathbf{m}))}{\phi_k(\mathbf{m})} \right) \mapsto \text{wt}_{\Gamma'}(\phi_k(\mathbf{m})),$$

so $\mathbf{m} \cdot \left(\frac{\text{wt}_{\Gamma'}(\phi_k(\mathbf{m}))}{\phi_k(\mathbf{m})} \right)$ is an element of weight ρ in Γ . From Lemma 3.4, the uniqueness is followed. Lemma 3.9 implies that Γ is connected as Γ' is connected.

To show (iii) in Definition 2.7, suppose that $\mathbf{p}' \cdot \mathbf{p} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{p}, \mathbf{p}' \in \overline{M}_{\geq 0}$. Note that

$$\phi_k(\mathbf{p}' \cdot \mathbf{p} \cdot \mathbf{m}_\rho) = \frac{\phi_k(\mathbf{p}' \cdot \mathbf{p} \cdot \mathbf{m}_\rho)}{\phi_k(\mathbf{p} \cdot \mathbf{m}_\rho)} \cdot \frac{\phi_k(\mathbf{p} \cdot \mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)} \cdot \phi_k(\mathbf{m}_\rho) \in \Gamma'.$$

Since Γ' is a G_k -brick, $\phi_k(\mathbf{p} \cdot \mathbf{m}_\rho) \in \Gamma'$. Thus $\mathbf{p} \cdot \mathbf{m}_\rho$ is in Γ . Therefore Γ is a G -prebrick.

To show that $S(\Gamma) = S(\Gamma')$, note that for $\mathbf{p} \in \overline{M}_{\geq 0}$ and $\mathbf{m}_\rho \in \Gamma$,

$$\frac{\mathbf{p} \cdot \mathbf{m}_\rho}{\text{wt}_\Gamma(\mathbf{p} \cdot \mathbf{m}_\rho)} = \frac{\phi_k(\mathbf{p} \cdot \mathbf{m}_\rho)}{\phi_k(\text{wt}_\Gamma(\mathbf{p} \cdot \mathbf{m}_\rho))} = \frac{\mathbf{n} \cdot \phi_k(\mathbf{m}_\rho)}{\text{wt}_{\Gamma'}(\mathbf{n} \cdot \phi_k(\mathbf{m}_\rho))} \in S(\Gamma')$$

where $\mathbf{n} = \frac{\phi_k(\mathbf{p} \cdot \mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)}$. Since $S(\Gamma)$ is generated by $\frac{\mathbf{p} \cdot \mathbf{m}_\rho}{\text{wt}_\Gamma(\mathbf{p} \cdot \mathbf{m}_\rho)}$, we proved that $S(\Gamma) \subset S(\Gamma')$.

For the opposite inclusion, suppose that $\mathbf{n} \in \Gamma'$. Let $\{\xi_j\}$ be the eigencoordinates with respect to the G_k -action. Lemma 3.9 shows that for every ξ_j there exist $x_i, \mathbf{m}_\rho \in \Gamma$ such that $\phi_k(x_i \cdot \mathbf{m}_\rho) = \xi_j \cdot \mathbf{n}$ with $\phi_k(\mathbf{m}_\rho) = \mathbf{n}$. Then

$$\frac{\xi_j \cdot \mathbf{n}}{\text{wt}_{\Gamma'}(\xi_j \cdot \mathbf{n})} = \frac{\phi_k(x_i \cdot \mathbf{m}_\rho)}{\phi_k(\text{wt}_{\Gamma'}(\phi_k(x_i \cdot \mathbf{m}_\rho)))} = \frac{\phi_k(x_i \cdot \mathbf{m}_\rho)}{\phi_k(\text{wt}_\Gamma(x_i \cdot \mathbf{m}_\rho))} = \frac{x_i \cdot \mathbf{m}_\rho}{\text{wt}_\Gamma(x_i \cdot \mathbf{m}_\rho)}.$$

This completes the proof. \square

Definition 3.11. The G -brick Γ in Proposition 3.10 is called the *natural inverse* of Γ' and denoted by $\phi_k^*(\Gamma')$.

3.2. Star subdivisions and bricksets. Let G be a finite diagonal group in $\text{GL}_n(\mathbb{C})$. Let X denote the quotient variety \mathbb{C}^n/G and X_v the toric variety given by the good star subdivision at $v = \frac{1}{r}(a_1, \dots, a_n)$. Recall that the toric fan of X_v contains the n -dimensional cone

$$\sigma_k := \text{Cone}(e_1, \dots, \hat{e}_k, v, \dots, e_n).$$

Note that X_v is covered by the affine toric open sets

$$U_k = \text{Spec } \mathbb{C}[\sigma_k^\vee \cap M] \cong \mathbb{C}^n/G_k,$$

where $G_k = L/L_k$.

Assume that Y is a normal toric variety admitting a proper birational morphism $Y \rightarrow X$. Let Σ denote the toric fan of Y . Assume further that there exists a dominant toric morphism $\overline{\varphi}: Y \rightarrow X_v$ fitting into the commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\overline{\varphi}} & X_v \\ & \searrow & \downarrow \\ & & X. \end{array}$$

As is standard in toric geometry (see eg. Section 3.3 in [3]), for each cone $\sigma \in \Sigma$, there exists a cone σ_k such that $\sigma \subset \sigma_k$. Therefore for each k , φ induces the following toric morphism

$$\overline{\varphi}_k: Y_k \rightarrow U_k,$$

where Y_k is the toric variety whose fan consists of the cones $\sigma \in \Sigma$ satisfying $\sigma \subset \sigma_k$. Note that since $X_v \rightarrow X$ is projective, if the morphism $Y \rightarrow X$ is projective, then so is φ_k .

Theorem 3.12. *With the assumption above, further assume that each $Y_k \rightarrow U_k$ has a G_k -brickset \mathfrak{S}_k . Define*

$$\mathfrak{S} := \bigcup_k \{ \phi_k^*(\Gamma') \mid \Gamma' \in \mathfrak{S}_k \}.$$

Then \mathfrak{S} is a G -brickset for the morphism $Y \rightarrow X$.

Proof. Let Σ_{\max} be the set of the n -dimensional cones in the fan Σ of Y . From Proposition 3.10, it follows that every object in the set \mathfrak{S} is a G -brick. It suffices to prove that the set \mathfrak{S} satisfies:

- (i) there exists a bijection $\Sigma_{\max} \rightarrow \mathfrak{S}$ sending σ to Γ_σ ;
- (ii) $S(\Gamma_\sigma) = \sigma^\vee \cap M$.

Let σ be an arbitrary n -dimensional cone in Σ . Then there exists a unique cone σ_k such that $\sigma \subset \sigma_k$. By the assumption, there is a unique G_k -brick $\Gamma' \in \mathfrak{S}_k$ such that $S(\Gamma') = \sigma_k^\vee \cap M$. Define

$$\Gamma_\sigma = \phi_k^*(\Gamma') := \{ \mathbf{m} \in \overline{M} \mid \phi_k(\mathbf{m}) \in \Gamma' \}.$$

By Proposition 3.10, we have

$$S(\Gamma_\sigma) = S(\Gamma') = \sigma_k^\vee \cap M,$$

and the proof is completed. \square

Please note that by Proposition 2.16, if there is $\theta \in \Theta$ satisfying that every Γ in \mathfrak{S} is θ -stable, then Y is isomorphic to Y_θ .

3.3. Star subdivisions and stability parameters. In this section, we discuss the existence of a stability parameter θ such that every G -brick in the brickset described in Theorem 3.12 is θ -stable.

Consider the good star subdivision at $v = \frac{1}{r}(a_1, \dots, a_n)$. For each k , let $\Theta^{(k)}$ be the GIT parameter space of G_k -constellations. Remember that by Lemma 3.4 we have the well-defined surjective map

$$\phi_k: G^\vee \rightarrow G_k^\vee$$

induced by the round down function ϕ_k . Note that the linear map

$$(\phi_k)_*: \Theta \rightarrow \Theta^{(k)}$$

defined by

$$(3.13) \quad [(\phi_k)_*(\theta)](\chi) = \sum_{\phi_k(\rho)=\chi} \theta(\rho) \text{ for } \chi \in G_k^\vee$$

is well-defined.

Let ρ_i denote the irreducible representation of G whose weight is i and χ_j the irreducible representation of G_k whose weight is j . First

note that Θ , which is a \mathbb{Q} -vector space of $(r-1)$ -dimension, has a \mathbb{Q} -basis $\{\theta_i \in \text{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \mid 1 \leq i < r\}$ where

$$(3.14) \quad \theta_i(\rho_l) = \begin{cases} 1 & \text{if } \rho_l = \rho_i, \\ -1 & \text{if } \rho_l \text{ is trivial,} \\ 0 & \text{otherwise,} \end{cases}$$

for $\rho_l \in G^\vee$. By the definition of the round down functions (see Lemma 3.4 or Remark 3.5), we have

$$[(\phi_k)_\star(\theta_i)](\chi_j) = \begin{cases} 1 & \text{if } j = i \pmod{a_k}, \\ -1 & \text{if } \chi_j \text{ is trivial,} \\ 0 & \text{otherwise,} \end{cases}$$

for $\chi_j \in G_k^\vee$. In particular, $(\phi_k)_\star(\theta_i) \equiv (\phi_k)_\star(\theta_{i'})$ if $i \equiv i' \pmod{a_k}$.

Remark 3.15. As is discussed above, $(\phi_k)_\star: \Theta \rightarrow \Theta^{(k)}$ is surjective. Indeed, $(\phi_k)_\star(\theta_i)$ for $1 \leq i < a$ form a \mathbb{Q} -basis of $\Theta^{(k)}$. \diamond

From now on, we only consider 3-dimensional cases. Consider the good star subdivision at $v = \frac{1}{r}(1, a, b)$ with $a < b$.

Lemma 3.16. *Consider the good star subdivision at $v = \frac{1}{r}(1, a, b)$ with $a < b$. Assume that a and b are coprime. Given $\theta^{(k)} \in \Theta^{(k)}$ for $k = 2, 3$, there exists $\theta_P \in \Theta$ such that*

$$(3.17) \quad (\phi_k)_\star(\theta) \equiv \theta^{(k)}$$

for all k .

Proof. Consider the linear map

$$\phi_\star = ((\phi_2)_\star, (\phi_3)_\star) : \Theta \rightarrow \Theta^{(2)} \oplus \Theta^{(3)}.$$

We need to prove that ϕ_\star is surjective. It suffices to show that

$$\{\phi_\star(\theta_i) \mid 1 \leq i \leq a+b-2\}$$

is linearly independent as the dimension of $\Theta^{(2)} \oplus \Theta^{(3)}$ is $a+b-2$. Using the fact that a and b are coprime, the assertion follows from a direct calculation. \square

4. MAIN THEOREM

4.1. Toric minimal model program. In this section, we recall the birational geometry of toric varieties (see [13]). Reid [13] introduced a combinatorial criterion for a toric variety to have terminal singularities and canonical singularities.

Theorem 4.1 (Reid [13]). *Let X be the toric variety corresponding to a fan Σ with a lattice L and the dual lattice M . Then X has only terminal singularities (resp. canonical singularities) if and only if any cone $\sigma \in \Sigma$ satisfies the conditions (i) and (ii) (resp. (i) and (iii)):*

- (i) *there exists an element $\mathbf{m} \in M_{\mathbb{Q}}$ such that $\langle \mathbf{u}, \mathbf{m} \rangle = 1$ for any primitive vector \mathbf{u} of σ ;*
- (ii) *there are no other lattice points in the set $\{\mathbf{u} \in \sigma \mid \langle \mathbf{u}, \mathbf{m} \rangle \leq 1\}$ except vertices;*
- (iii) *there are no other lattice points in the set $\{\mathbf{u} \in \sigma \mid \langle \mathbf{u}, \mathbf{m} \rangle < 1\}$ except the origin.*

Theorem 4.2 (Reid [13]). *Let X be a quasiprojective toric variety and $V \rightarrow X$ a projective birational toric morphism with V smooth. Then there exists the following diagram*

$$\begin{array}{ccccc}
 V & \dashrightarrow & Y & \xrightarrow{\overline{\varphi}} & X_{\text{can}} \\
 & \searrow & \downarrow \varphi & \swarrow \nu & \\
 & & X & &
 \end{array}$$

where

- (i) *X_{can} has canonical singularities, $\nu: X_{\text{can}} \rightarrow X$ is a projective birational morphism, and $K_{X_{\text{can}}}$ is ν -ample;*
- (ii) *Y has \mathbb{Q} -factorial terminal singularities, $\varphi: Y \rightarrow X$ is a projective birational morphism, and K_Y is φ -nef, i.e. $\overline{\varphi}$ is crepant.*

Definition 4.3. In Theorem 4.2, we say that:

- (i) the variety X_{can} is a *relative canonical model* of X ;
- (ii) the variety Y is a *relative minimal model* of X .

Convention 4.4. In this article, relative minimal models of X are always projective over X .

4.2. The Craw–Ishii conjecture. For a finite abelian subgroup G of $\text{SL}_3(\mathbb{C})$, Craw and Ishii proved that every projective crepant resolution of \mathbb{C}^3/G is isomorphic to \mathcal{M}_{θ} for a suitable parameter θ .

Theorem 4.5 (Craw–Ishii [2]). *For a finite abelian subgroup G of $\text{SL}_3(\mathbb{C})$, let Y be a relative minimal model of \mathbb{C}^3/G . Then Y is isomorphic to \mathcal{M}_{θ} for a suitable θ .*

They conjectured that the same holds without the abelian assumption. We further conjecture that the same is true for all finite group $G \subset \text{GL}_3(\mathbb{C})$ if Y is a smooth relative minimal model.

Conjecture 4.6 (Craw–Ishii conjecture). *For a finite subgroup G of $\text{GL}_3(\mathbb{C})$, let Y be a relative minimal model of \mathbb{C}^3/G . If Y is smooth, then Y is isomorphic to (the birational component Y_{θ} of) \mathcal{M}_{θ} for a suitable θ .*

To prove the theorem above, Craw and Ishii showed that a flop of \mathcal{M}_{θ} is isomorphic to $\mathcal{M}_{\theta'}$ for some parameter θ' as two crepant resolutions are connected by a sequence of flops. This completes the proof because

we already knew that $G\text{-Hilb } \mathbb{C}^3$ is a crepant resolution of \mathbb{C}^3/G by Bridgeland–King–Reid [1] for $G \subset \mathrm{SL}_3(\mathbb{C})$.

Note that for $G \not\subset \mathrm{SL}_3(\mathbb{C})$ we do not have a moduli description of any relative minimal model of \mathbb{C}^3/G yet.

Remark 4.7. From Theorem 4.5, we have a simple corollary as follows. For G a finite abelian subgroup in $\mathrm{SL}_3(\mathbb{C})$ and Y a projective crepant resolution of $X = \mathbb{C}^3/G$, there exist:

- (i) a G -brickset \mathfrak{S} for $Y \rightarrow X$;
- (ii) a stability parameter θ such that $\Gamma \in \mathfrak{S}$ is θ -stable.

We use this to prove the Craw–Ishii conjecture for some cases. \diamond

4.3. The first case: $r = abc + a + b + 1$. In this section, as the first example, we prove the Craw–Ishii conjecture for the group G of type $\frac{1}{r}(1, a, b)$ with $r = abc + a + b + 1$ where a, b, c are positive integers with b coprime to a . Consider the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, b),$$

and the cone $\sigma_+ = \mathrm{Cone}(e_1, e_2, e_3)$. The lattice point $v := \frac{1}{r}(1, a, b)$ is in the interior of the simplex Δ where

$$\Delta := \{\mathbf{u} \in \sigma_+ \mid \langle \mathbf{u}, x_1 x_2 x_3 \rangle \leq 1\},$$

with considering the monomial $x_1 x_2 x_3$ as an element in $M_{\mathbb{Q}}$. Thus the quotient singularity $X := \mathbb{C}^3/G$ defined by the cone σ_+ is not a canonical singularity.

Consider the star subdivision of σ_+ at v . Let X_v denote the toric variety corresponding to the star subdivision. From Section 3.1, we have:

- (i) the cone $\sigma_1 = \mathrm{Cone}(v, e_2, e_3)$ defines a smooth open set U_1 ;
- (ii) the cone $\sigma_2 = \mathrm{Cone}(e_1, v, e_3)$ defines the affine toric variety $U_2 = \mathbb{C}^3/G_2$ with G_2 of type $\frac{1}{a}(1, -r, b)$;
- (iii) the cone $\sigma_3 = \mathrm{Cone}(e_1, e_2, v)$ defines the affine toric variety $U_3 = \mathbb{C}^3/G_3$ with G_3 of type $\frac{1}{b}(1, a, -r)$.

Note that σ_2 and σ_3 define Gorenstein 3-fold abelian quotient singularities. Hence the star subdivision at v has only canonical singularities. Since a star subdivision induces a projective toric morphism, from the ramification formula (3.1), it follows that the star subdivision of σ_+ at v defines the relative canonical model X_{can} of X , i.e. X_v is the relative canonical model of X .

Suppose that $\varphi: Y \rightarrow X$ is a relative minimal model. Then there exists a projective crepant morphism $\overline{\varphi}: Y \rightarrow X_v$ fitting into the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\overline{\varphi}} & X_v \\ & \searrow \varphi & \downarrow \\ & & X. \end{array}$$

As is discussed in Section 3.2, we have the following three induced projective crepant morphisms:

- (i) $\overline{\varphi}_1: Y_1 \rightarrow U_1 = \mathbb{C}^3$;
- (ii) $\overline{\varphi}_2: Y_2 \rightarrow U_2 = \mathbb{C}^3/G_2$;
- (iii) $\overline{\varphi}_3: Y_3 \rightarrow U_3 = \mathbb{C}^3/G_3$.

Here Y_k denotes the toric variety given by the cones σ such that $\sigma \subset \sigma_k$. Note that $\overline{\varphi}_1$ is an isomorphism. From the Craw–Ishii Theorem [2], it follows that there exists a generic GIT parameter $\theta^{(k)}$ such that Y_k is the moduli space of $\theta^{(k)}$ -stable G_k -constellations. Thus we have:

- (i) a G_k -brickset \mathfrak{S}_k for $Y_k \rightarrow U_k$;
- (ii) a stability parameter $\theta^{(k)}$ such that $\Gamma \in \mathfrak{S}_k$ is $\theta^{(k)}$ -stable.

By Theorem 3.12, there exists a G -brickset \mathfrak{S} for $Y \rightarrow X$. Note that to the cone σ_1 , we assign the G -brick

$$\Gamma := \phi_1^{-1}(\mathbf{1}) = \{1, x_1, x_1^2, \dots, x_1^{r-1}\},$$

which satisfies $S(\Gamma) = \sigma_1^\vee \cap M$.

Now we show that there exists a parameter θ such that every $\Gamma \in \mathfrak{S}$ is θ -stable. First note that since a and b are coprime, by Lemma 3.16, there exists $\theta_P \in \Theta$ satisfies (3.17) for given $\theta^{(2)}$ and $\theta^{(3)}$.

Define the GIT parameter $\psi \in \Theta$ by

$$(4.8) \quad \psi(\rho) = \begin{cases} -1 & \text{if } 0 \leq \text{wt}(\rho) < b, \\ -1 & \text{if } \text{wt}(\rho) = a + b, \\ 1 & \text{if } r - b - 1 \leq \text{wt}(\rho) < r, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.9. The parameter ψ in (4.8) has the following properties for each k .

- (i) For any $\chi \in G_k^\vee$, we have $[(\phi_k)_*(\psi)](\chi) \equiv 0$, i.e.

$$\sum_{i=j \pmod{a_k}} \psi(\rho_i) = 0$$

for each $0 \leq j < a_k$.

- (ii) For any i with $0 \leq i < a_k$, we have $\psi(\rho_i) < 0$.
- (iii) For a monomial \mathbf{m} of weight i with $a_k \leq i < r$, define

$$A := A(\mathbf{m}) := \{x_k^l \cdot \mathbf{m} \mid \phi_k(x_k^l \cdot \mathbf{m}) = \phi_k(\mathbf{m}) \text{ for some } l \geq 0\}.$$

Then we have $\psi(A) > 0$.

These are the key properties we use for the existence of a suitable parameter. \diamond

Proposition 4.10. *For a sufficiently large natural number m , set*

$$(4.11) \quad \theta := \theta_P + m\psi,$$

with θ_P satisfying (3.17). If a G -brick Γ is in \mathfrak{S} described above, then Γ is θ -stable.

Proof. Let Γ be a G -brick in \mathfrak{S} and σ the corresponding cone.

If the cone σ is contained in σ_1 , then the corresponding G -graph is $\Gamma = \{1, x_1, x_1^2, \dots, x_1^{r-2}, x_1^{r-1}\}$. Note that any nonzero proper submodule \mathcal{G} of $C(\Gamma)$ is given by

$$A = \{x_1^j, x_1^{j+1}, \dots, x_1^{r-2}, x_1^{r-1}\}$$

for some $1 \leq j \leq r-1$ by Lemma 2.9. Thus $\psi(\mathcal{G}) > 0$ by definition. From this, it follows that Γ is θ -stable for sufficiently large m .

For the other cases, assume that Γ is the G -brick corresponding to a cone $\sigma \subset \sigma_k$. Let Γ' be the G_k -brick corresponding to Γ and \mathcal{G} a nonzero proper submodule of $C(\Gamma)$ with \mathbb{C} -basis $A \subset \Gamma$. Recall that

$$\Gamma = \{\mathbf{m} \in \overline{M} \mid \phi_k(\mathbf{m}) \in \Gamma'\},$$

as in Proposition 3.10. We have the following two cases:

- (i) $A = \phi_k^{-1}(\phi_k(A)) := \{\mathbf{m} \in \overline{M} \mid \phi_k(\mathbf{m}) \in \phi_k(A)\};$
- (ii) $A \subsetneq \phi_k^{-1}(\phi_k(A)).$

In case (i), $\psi(\mathcal{G}) = 0$ by definition. Moreover, we can see that the set $\phi_k(A)$ defines a nonzero proper submodule \mathcal{G}' of $C(\Gamma')$ as follows. Let ξ_1, ξ_2, ξ_3 be the eigencoordinates with respect to the G_k -action on \mathbb{C}^3 . Suppose that $\xi_j \cdot \phi_k(\mathbf{m}_\rho) \in \Gamma'$ for some $\mathbf{m}_\rho \in A$. Lemma 3.9 implies that there exist $\mathbf{m}_{\rho'}$ and x_i such that

$$\phi_k(x_i \cdot \mathbf{m}_{\rho'}) = \xi_j \cdot \phi_k(\mathbf{m}_\rho) \quad \text{with} \quad \phi_k(\mathbf{m}_{\rho'}) = \phi_k(\mathbf{m}_\rho).$$

As A is a \mathbb{C} -basis of \mathcal{G} , Lemma 2.9 implies that $x_i \cdot \mathbf{m}_{\rho'} \in A$. Thus $\xi_j \cdot \phi_k(\mathbf{m}_\rho)$ is in $\phi_k(A)$. This shows that $\phi_k(A)$ is a \mathbb{C} -basis of a nonzero proper submodule \mathcal{G}' of $C(\Gamma')$. Since

$$(\phi_k)_*(\theta) \equiv \theta^{(k)},$$

we have $\theta(\mathcal{G}) = \theta^{(k)}(\mathcal{G}') > 0$ as \mathcal{G}' is a submodule of the $\theta^{(k)}$ -stable constellation $C(\Gamma')$.

Consider case (ii). Observe that

$$\sum_{\phi_k(\rho') \in \phi_k(A)} \psi(\rho') = 0$$

by the definition of ψ . Lemma 3.6 implies that if \mathbf{m}_ρ in $\phi_k^{-1}(\phi_k(A)) \setminus A$, then $0 \leq \text{wt}(\rho) < r - b$. Moreover we have

$$\sum_{\rho' \in \phi_k^{-1}(\phi_k(A)) \setminus A} \psi(\rho') < 0.$$

Thus $\psi(\mathcal{G}) > 0$. Therefore $\theta(\mathcal{G}) > 0$ for sufficiently large m .

Since there exist a finite number of G -bricks in \mathfrak{S} , we are done. \square

Remark 4.12. The parameter θ in Proposition 4.10 does not need to be generic. However, since the condition for θ is an open condition, there exists a generic parameter in a small neighbourhood of θ . \diamond

As we have proved the existence of a suitable generic parameter θ , we have the following theorem.

Theorem 4.13. *For positive integers a, b, c with b coprime to a , let G be the group of type $\frac{1}{r}(1, a, b)$ with $r = abc + a + b + 1$. Assume that $Y \rightarrow X := \mathbb{C}^3/G$ is any relative minimal model of X . Then Y is isomorphic to the birational component Y_θ of the moduli space \mathcal{M}_θ of θ -stable G -constellations for a suitable parameter θ .*

In some small cases (eg. $\frac{1}{20}(1, 3, 4)$), the irreducible component Y_θ is actually a connected component of \mathcal{M}_θ . However, we do not have a big example such that \mathcal{M}_θ itself is irreducible.

Question 4.14. In the situation as above, is \mathcal{M}_θ in the theorem above irreducible?

Example 4.15. Let G be the group of type $\frac{1}{20}(1, 3, 4)$ as in Example 3.2. Consider the star subdivision at $v = \frac{1}{20}(1, 3, 4)$. Then the star subdivision gives the relative canonical model of $X = \mathbb{C}^3/G$.

Let $\phi: Y \rightarrow X$ be a relative minimal model whose fan is shown in Figure 4.1.

There exist the two induced projective crepant resolutions:

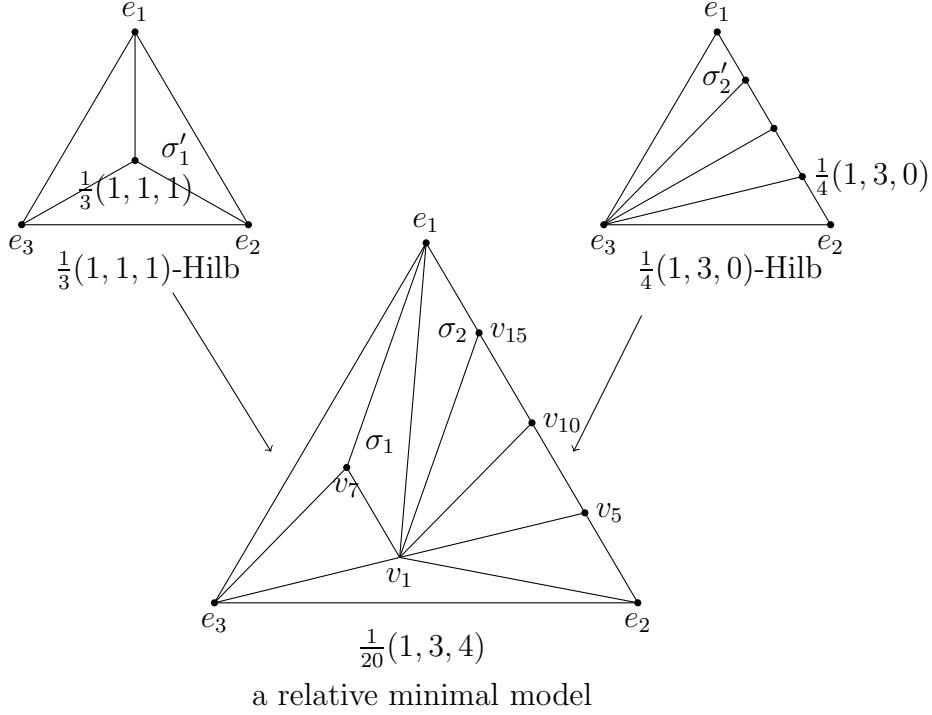
- (i) $\overline{\varphi}_2: Y_2 \rightarrow U_2 = \mathbb{C}^3/G_2$;
- (ii) $\overline{\varphi}_3: Y_3 \rightarrow U_3 = \mathbb{C}^3/G_3$.

Here G_2 is of type $\frac{1}{3}(1, 1, 1)$ and G_3 is of type $\frac{1}{4}(1, 3, 0)$. Note that Y_2 and Y_3 are G_2 -Hilb \mathbb{C}^3 and G_3 -Hilb \mathbb{C}^3 , respectively.

We illustrate how to calculate G -bricks associated to the following cones:

$$\begin{aligned} \sigma_1 &:= \text{Cone}\left((1, 0, 0), \frac{1}{20}(1, 3, 4), \frac{1}{20}(7, 1, 8)\right), \\ \sigma_2 &:= \text{Cone}\left((1, 0, 0), \frac{1}{20}(1, 3, 4), \frac{1}{20}(15, 5, 0)\right). \end{aligned}$$

Note that the cone σ_1 is in $\text{Cone}(e_1, v_1, e_3)$. Moreover, observe that the left fan corresponds to G_2 -Hilb \mathbb{C}^3 with G_2 of type $\frac{1}{3}(1, 1, 1)$. Consider the cone σ'_1 in the fan of G_2 -Hilb \mathbb{C}^3 corresponding to σ_1 . Let

FIGURE 4.1. Recursion process for $\frac{1}{20}(1, 3, 4)$

ξ, η, ζ denote the eigencoordinates for G_2 . The corresponding G_2 -brick is

$$\Gamma'_1 = \{1, \zeta, \zeta^2\}.$$

The G -brick Γ_1 corresponding to σ_1 is

$$\Gamma_1 \stackrel{\text{def}}{=} \{x^{m_1}y^{m_2}z^{m_3} \in \overline{M} \mid \phi_2(x^{m_1}y^{m_2}z^{m_3}) \in \Gamma'_1\}$$

where the left round down function ϕ_2 is defined by

$$\phi_2(x^{m_1}y^{m_2}z^{m_3}) = \xi^{m_1}\eta^{\lfloor \frac{1}{20}m_1 + \frac{3}{20}m_2 + \frac{4}{20}m_3 \rfloor} \zeta^{m_3}.$$

Thus

$$\Gamma_1 = \left\{ \begin{array}{ccccccccc} y^{-2}z^2 & y^{-1}z^2 & z^2 & yz^2 & y^2z^2 & y^3z^2 & & & \\ & y^{-1}z & z & yz & y^2z & y^3z & y^4z & y^5z & \\ & & 1 & y & y^2 & y^3 & y^4 & y^5 & y^6 \end{array} \right\}.$$

Observe that the cone σ_2 is in $\text{Cone}(e_1, e_2, v_1)$. The right fan is the fan of $G_3\text{-Hilb } \mathbb{C}^3$, where G_3 is of type $\frac{1}{4}(1, 3, 0)$. Let α, β, γ be the eigencoordinates. For the cone σ'_2 corresponding to σ_2 , observe that the corresponding G_3 -brick is

$$\Gamma'_2 = \{1, \beta, \beta^2, \beta^3\}.$$

The G -brick Γ_2 corresponding to σ_2 is

$$\Gamma_2 \stackrel{\text{def}}{=} \{x^{m_1}y^{m_2}z^{m_3} \in \overline{M} \mid \phi_3(x^{m_1}y^{m_2}z^{m_3}) \in \Gamma'_2\}$$

where the right round down function ϕ_3 is

$$\phi_3(x^{m_1}y^{m_2}z^{m_3}) = \alpha^{m_1}\beta^{m_2}\gamma^{\lfloor \frac{1}{20}m_1 + \frac{3}{20}m_2 + \frac{4}{20}m_3 \rfloor}.$$

Thus

$$\Gamma_2 = \begin{Bmatrix} y^3z^{-2} & y^3z^{-1} & y^3 & y^3z & y^3z^2 & & \\ & y^2z^{-1} & y^2 & y^2z & y^2z^2 & y^2z^3 & \\ & & y & yz & yz^2 & yz^3 & yz^4 \\ & & 1 & z & z^2 & z^3 & z^4 \end{Bmatrix}.$$

Note that $S(\Gamma_1) = \sigma_1^\vee \cap M$ and $S(\Gamma_2) = \sigma_2^\vee \cap M$.

Now we turn to stability parameters. Since Y_k is G_k -Hilb for each $k = 2, 3$, from (2.4) we can take

$$\theta^{(2)} = (-2, 1, 1), \quad \theta^{(3)} = (-3, 1, 1, 1).$$

Then the condition (3.17) of θ_P for given $\theta^{(2)}$ and $\theta^{(3)}$ is

$$\begin{cases} -2 &= \sum_{l=0}^6 \theta_P(\rho_{3l}), \\ 1 &= \sum_{l=0}^6 \theta_P(\rho_{3l+1}), \\ 1 &= \sum_{l=0}^5 \theta_P(\rho_{3l+2}), \\ -3 &= \sum_{l=0}^4 \theta_P(\rho_{4l}), \\ 1 &= \sum_{l=0}^4 \theta_P(\rho_{4l+1}), \\ 1 &= \sum_{l=0}^4 \theta_P(\rho_{4l+2}), \\ 1 &= \sum_{l=0}^4 \theta_P(\rho_{4l+3}). \end{cases}$$

Take

$$\theta_P = (-3, 0, 0, 0, 0, 1, 1, 1, 0, \dots, 0)$$

as a solution of the equations above. For ψ in (4.8), define $\theta = \theta_P + m\psi$:

$$\theta(\rho_i) = (\theta_P + m\psi)(\rho_i) = \begin{cases} -3 - m & \text{if } i = 0, \\ -m & \text{if } 1 \leq i \leq 3, \\ 0 & \text{if } i = 4, \\ 1 & \text{if } i = 5 \text{ or } 6, \\ 1 - m & \text{if } i = 7, \\ m & \text{if } 15 \leq i \leq 19, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the G -brick Γ_2 above:

$$\Gamma_2 = \begin{Bmatrix} y^3z^{-2} & y^3z^{-1} & y^3 & y^3z & y^3z^2 & & \\ & y^2z^{-1} & y^2 & y^2z & y^2z^2 & y^2z^3 & \\ & & y & yz & yz^2 & yz^3 & yz^4 \\ & & 1 & z & z^2 & z^3 & z^4 \end{Bmatrix}.$$

As examples, consider the two submodules \mathcal{G} , \mathcal{H} generated by A and B , respectively, where

$$A = \left\{ \begin{matrix} y^3 z^{-2} & y^3 z^{-1} & y^3 & y^3 z & y^3 z^2 \\ & y^2 z^{-1} & y^2 & y^2 z & y^2 z^2 & y^2 z^3 \end{matrix} \right\},$$

$$B = \left\{ \begin{matrix} y^3 & y^3 z & y^3 z^2 \\ y^2 & y^2 z & y^2 z^2 & y^2 z^3 \\ y & yz & yz^2 & yz^3 & yz^4 \\ 1 & z & z^2 & z^3 & z^4 \end{matrix} \right\}.$$

First consider the submodule \mathcal{G} . Note that $\psi(\mathcal{G}) = 0$. By definition, note that $\phi_3(A) = \{\beta^2, \beta^3\}$ forms a basis of a submodule \mathcal{G}' of $C(\Gamma'_2)$ with $\theta(\mathcal{G}) = \theta^{(3)}(\mathcal{G}')$. Thus

$$\theta(\mathcal{G}) = \theta^{(3)}(\mathcal{G}') = 2 > 0.$$

For the submodule \mathcal{H} , note that $\phi_3^{-1}(\phi_3(B))$ contains $y^2 z^{-1}$, $y^3 z^{-1}$ and $y^3 z^{-2}$. Observe that $\psi(\mathcal{H}) > 0$. Thus $\theta(\mathcal{H})$ is positive for large enough m . More precisely,

$$\theta(\mathcal{H}) = -3 + 1 + 1 + m + m = 2m - 1$$

is positive if $m > \frac{1}{2}$. ◇

4.4. The second case: $r = abc + a - 2b + 1$. Consider the group of type $\frac{1}{r}(1, a, b)$. Assume that the star subdivision at $v = \frac{1}{r}(1, a, b)$ gives:

- (i) $\sigma_2 := \text{Cone}(e_1, v, e_3)$ is of type $\frac{1}{a}(1, 1, 1)$ for $a \geq 4$;
- (ii) $\sigma_3 := \text{Cone}(e_1, e_2, v)$ is a Gorenstein quotient singularity.

This means that:

- (i) $-r \equiv 1 \pmod{a}$;
- (ii) $1 - r + b \equiv 3 \pmod{a}$;
- (iii) $1 - r + a \equiv 0 \pmod{b}$.

In the rest of this section, we consider the case where

$$r = abc - 2b + a + 1 \quad \text{with} \quad b = ak + 1, a \geq 4$$

for some positive integers c, k . Consider the lattice

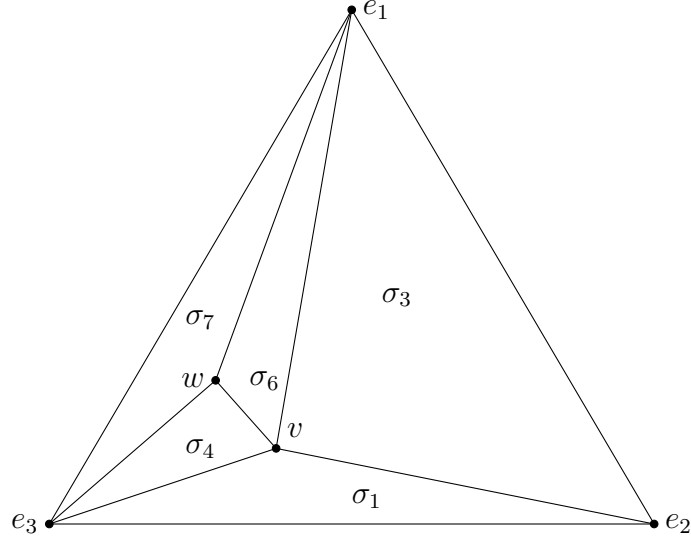
$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, b),$$

Let v and w denote the lattice points

$$v := \frac{1}{r}(1, a, b) \quad \text{and} \quad w := \frac{1}{r}\left(\frac{r+1}{a}, 1, \frac{r+b}{a}\right).$$

Let X_v denote the toric variety corresponding to the star subdivision at v . In this case, X_v is not the relative canonical model of $X = \mathbb{C}^3/G$ because the quotient of type $\frac{1}{a}(1, 1, 1)$ is not canonical for $a \geq 4$. The relative canonical model depends on c . We have the two cases:

- (a) $c \geq 2$;

FIGURE 4.2. Canonical model for $c \geq 2$ (b) $c = 1$.

Case (a): $c \geq 2$. Consider the case where $c \geq 2$. In this case, the relative canonical model is given by the fan consisting of the following five cones and their faces:

$$\begin{aligned} \sigma_1 &= \text{Cone}(v, e_2, e_3), & \sigma_3 &= \text{Cone}(e_1, e_2, v), \\ \sigma_4 &= \text{Cone}(w, v, e_3), & \sigma_6 &= \text{Cone}(e_1, v, w), & \sigma_7 &= \text{Cone}(e_1, w, e_3). \end{aligned}$$

Indeed, the cone σ_2 defines a Gorenstein quotient singularity and the others define smooth affine toric open sets. We can check directly $K_{X_{\text{can}}}$ is ample over X .

Since there exists a projective morphism $X_{\text{can}} \rightarrow X_v$, for every relative minimal model $\varphi: Y \rightarrow X$, we have a projective morphism $\bar{\varphi}: Y \rightarrow X_v$ fitting into:

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & X_{\text{can}} \\ & \searrow \bar{\varphi} & \downarrow \\ & & X_v \\ & \searrow \varphi & \downarrow \nu \\ & & X. \end{array}$$

The morphism $\bar{\varphi}$ induces two projective morphisms:

- (i) $\bar{\varphi}_2: Y_2 \rightarrow U_2 = \mathbb{C}^3/G_2$;
- (ii) $\bar{\varphi}_3: Y_3 \rightarrow U_3 = \mathbb{C}^3/G_3$.

As is seen above, G_2 is of type $\frac{1}{a}(1, 1, 1)$ and the induced morphism $\overline{\varphi}_2$ is given by

$$G_2\text{-Hilb } \mathbb{C}^3 \rightarrow \mathbb{C}^3/G_2$$

where G_2 is of type $\frac{1}{a}(1, 1, 1)$. Thus it follows that there exist a brickset \mathfrak{S}_2 for $Y_2 \rightarrow U_2$ and $\theta^{(2)}$ for the brickset \mathfrak{S}_2 . On the other hand, since U_3 is a Gorenstein quotient singularity, by the Craw–Ishii Theorem [2], there exists a brickset \mathfrak{S}_3 for $Y_3 \rightarrow U_3$ and $\theta^{(3)}$ for the brickset \mathfrak{S}_3 .

From Theorem 3.12, there is a G -brickset for $Y \rightarrow X$. Now it suffices to find a GIT parameter θ such that every $\Gamma \in \mathfrak{S}$ is θ -stable. Define the GIT parameter $\psi \in \Theta$ by

$$\psi(\rho) = \begin{cases} -1 & \text{if } 0 \leq \text{wt}(\rho) < b, \\ -1 & \text{if } \text{wt}(\rho) = 2ab - 5b + 3, \\ 1 & \text{if } \text{wt}(\rho) = r - a - b + 2, \\ 1 & \text{if } r - b \leq \text{wt}(\rho) < r, \\ 0 & \text{otherwise.} \end{cases}$$

Note that ψ above has the same properties in Remark 4.9. Thus the same proof works for the existence of θ as in Proposition 4.10. Therefore the following theorem follows.

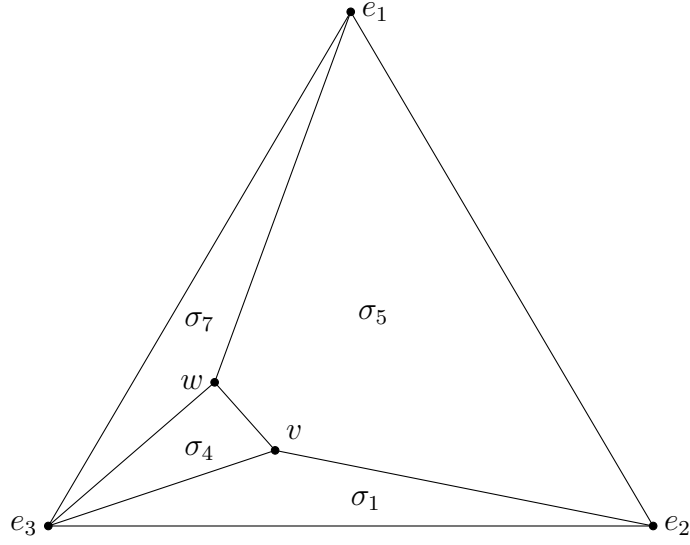
Theorem 4.16. *Consider positive integers a, k, c with $c \geq 2$, $a \geq 4$ and $b = ak + 1$. Let G be the group of type $\frac{1}{r}(1, a, b)$ with $r = abc + a - 2b + 1$. Let $Y \rightarrow X := \mathbb{C}^3/G$ be a relative minimal model of X . Then Y is isomorphic to the birational component Y_θ of the moduli space \mathcal{M}_θ of θ -stable G -constellations for a suitable parameter θ .*

Case (b): $c = 1$. For the case where $c = 1$, the fan of the relative canonical model consists of the following four cones and their faces:

$$\begin{aligned} \sigma_1 &= \text{Cone}(v, e_2, e_3), & \sigma_5 &= \text{Cone}(e_1, e_2, v, w), \\ \sigma_4 &= \text{Cone}(w, v, e_3), & \sigma_7 &= \text{Cone}(e_1, w, e_3). \end{aligned}$$

Indeed, the cone σ_5 defines a toric Gorenstein singularity and hence it is canonical. Note that since the cone σ_5 is not simplicial, the corresponding affine toric variety is not a quotient type. In particular, the relative canonical model does not need to be obtained by a sequence of star subdivisions because a star subdivision of a simplicial fan is simplicial.

Note that since X_{can} is Gorenstein, every relative minimal model is smooth. However, some relative minimal of X does not have a morphism to X_v (See Example 4.18). For a relative minimal model Y admitting a morphism to X_v and for $a \geq 6$, we can prove in the same

FIGURE 4.3. Canonical model for $c = 1$

way as in Case (a) with the following ψ :

$$\psi(\rho) = \begin{cases} -1 & \text{if } 0 \leq \text{wt}(\rho) < b, \\ -1 & \text{if } \text{wt}(\rho) = ab - 5b + 3, \\ 1 & \text{if } \text{wt}(\rho) = r - a - b + 2, \\ 1 & \text{if } r - b \leq \text{wt}(\rho) < r, \\ 0 & \text{otherwise.} \end{cases}$$

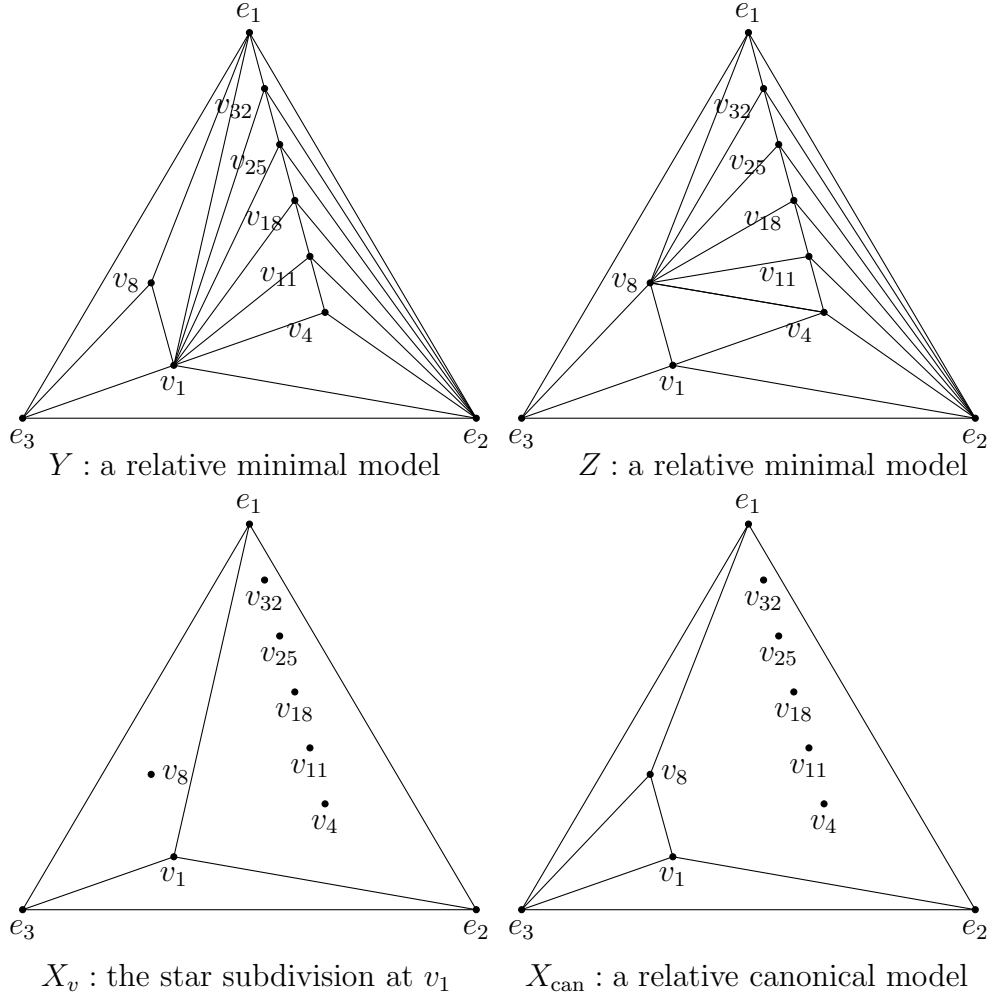
Proposition 4.17. *For positive integers a, k , let G be the group of type $\frac{1}{r}(1, a, b)$ with $r = ab + a - 2b + 1$ and $b = ak + 1$. Furthermore assume that $a \geq 6$. Let X_v denote the toric variety given by the star subdivision of σ_+ at $v = \frac{1}{r}(1, a, b)$. Let $Y \rightarrow X := \mathbb{C}^3/G$ be a relative minimal model admitting a morphism $Y \rightarrow X_v$. Then Y is isomorphic to the birational component Y_θ of the moduli space \mathcal{M}_θ of θ -stable G -constellations for a suitable parameter θ .*

Example 4.18. Consider the group G of type $\frac{1}{39}(1, 5, 11)$. Then the star subdivision at $v = \frac{1}{39}(1, 5, 11)$ gives:

- (i) $\sigma_2 := \text{Cone}(e_1, v, e_3)$ corresponds to the quotient singularity of type $\frac{1}{5}(1, 1, 1)$;
- (ii) $\sigma_3 := \text{Cone}(e_1, e_2, v)$ corresponds to the quotient singularity of type $\frac{1}{11}(1, 5, 5)$.

As is discussed above, the relative canonical model X_{can} of $X = \mathbb{C}^3/G$ is Gorenstein, but not \mathbb{Q} -factorial.

Let v_i denote the lattice point $\frac{1}{r}(\bar{i}, \overline{5i}, \overline{11i})$ where $\bar{\cdot}$ denotes the residue modulo r . In particular, $v_1 = v$ and $v_8 = w$. Note that there exists a plane Π containing e_1, e_2, v_1 and v_8 . Observe that the lattice

FIGURE 4.4. Fans of birational models for $\frac{1}{39}(1, 5, 11)$

points v_4 , v_{11} , v_{18} , v_{25} , and v_{32} lie on the plane Π . Thus subdividing the cone σ_5 into smooth cones only using these points defines a crepant resolution of the toric singularity given by σ_5 where

$$\sigma_5 = \text{Cone}(e_1, e_2, v_1, v_8).$$

In Figure 4.4, the variety Y is a relative minimal model of X admitting a morphism to X_v . Actually one can prove that Y is isomorphic to Y_θ for some θ . On the other hand, the variety Z is a relative minimal model having no morphism to X_v . At this moment, we do not know that whether Z is isomorphic to Y_θ for some θ . \diamond

4.5. Discussions.

4.5.1. *Smoothness of minimal models.* From Theorem 4.2, it follows that if the relative canonical model X_{can} is Gorenstein, then any relative minimal model is Gorenstein. Since a toric Gorenstein 3-fold terminal singularity is smooth, every relative minimal model is smooth if the relative canonical model X_{can} is Gorenstein for G being abelian. However, we do not know any sufficient condition for the group G of type $\frac{1}{r}(1, a, b)$ having the Gorenstein relative canonical model X_{can} of \mathbb{C}^3/G .

Question 4.19. Let G be the group of type $\frac{1}{r}(1, a, b)$ with $a+b+1 < r$. Let X be the quotient \mathbb{C}^3/G and X_{can} the relative canonical model of X . When is X_{can} Gorenstein? If so, when can we obtain X_{can} by a sequence of star subdivisions?

4.5.2. *Other stability parameters.* The theorems above said that for a relative minimal model Y there exists some parameter θ such that Y_θ is isomorphic to Y . We can ask whether Y_θ is a relative minimal model for all generic θ or not.

Sara Muhvić calculated the following:

- (i) for the type of $\frac{1}{12}(1, 2, 3)$, $G\text{-Hilb } \mathbb{C}^3$ is smooth but not a relative minimal model;
- (ii) for the type of $\frac{1}{24}(1, 3, 5)$, $G\text{-Hilb } \mathbb{C}^3$ is not even smooth.

Thus it seems that there exist few chambers in Θ giving a relative minimal model of \mathbb{C}^3/G .

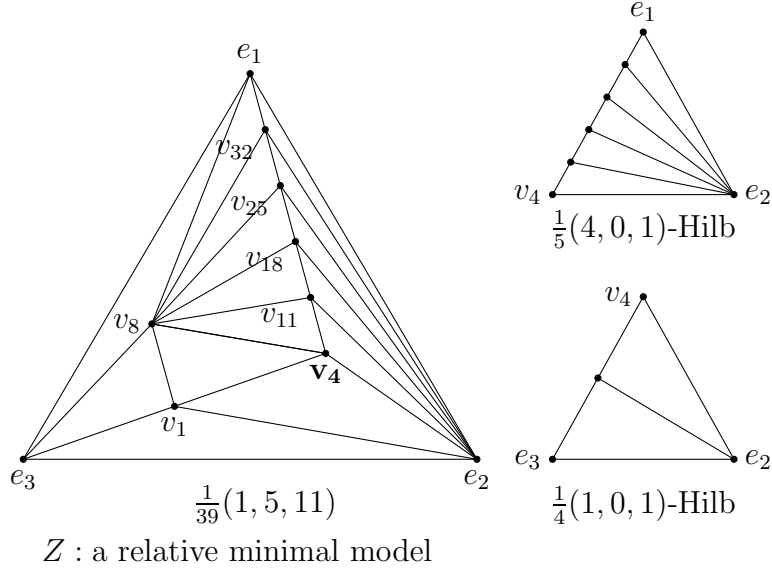
Question 4.20. Let G be the group of type in Section 4.3 or Section 4.4. For which θ , is Y_θ a relative minimal model?

4.5.3. *Existence of stability parameters.* Let $Y \rightarrow X = \mathbb{C}^3/G$ be a relative minimal model admitting a morphism to X_v where X_v is the toric variety given by the star subdivision of σ_+ at v . The main theorem was proved by showing the three statements:

- (i) there exists a G -brickset \mathfrak{S} for $Y \rightarrow X$ using round down functions;
- (ii) the linear map $\Theta \rightarrow \Theta^{(1)} \oplus \Theta^{(2)} \oplus \Theta^{(3)}$ is surjective in (3.17);
- (iii) there exists a stability parameter ψ satisfying (4.9).

To prove (i), we only used the existence of a G_k -brickset for G_k , whose order is smaller than that of G . When we proved (ii), we only use the assumption that a and b are coprime. However, showing the existence of ψ in (iii) was done on a case by case basis in Section 4.3 and Section 4.4. It would be interesting if we have a systematic way to produce such a parameter ψ .

Question 4.21. Is there a systematic method to find a stability parameter ψ satisfying the properties in Remark 4.9 for a star subdivision?

FIGURE A.1. Recursion process for Z APPENDIX A. $\frac{1}{39}(1, 5, 11)$ TYPE

Let G be the group of type $\frac{1}{39}(1, 5, 11)$ as in Example 4.18. Consider the relative minimal model Z in Figure 4.4. In this section, although we cannot see that Z is isomorphic to the birational component Y_θ of \mathcal{M}_θ , we show that there exists a G -brickset for $Z \rightarrow X = \mathbb{C}^3/G$.

Although there is no morphism $Z \rightarrow X_v$, there exists a morphism $\bar{\varphi}: Z \rightarrow X_u$ where X_u is the toric variety given by the star subdivision at $u = v_4 = \frac{1}{39}(4, 20, 5)$. The star subdivision of σ_+ at u produces the three cones:

$$\sigma_1 = \text{Cone}(u, e_2, e_3), \quad \sigma_2 = \text{Cone}(e_1, e_2, u), \quad \sigma_3 = \text{Cone}(e_1, u, e_3).$$

The morphism $\bar{\varphi}$ induces the following three morphisms:

- (i) $\bar{\varphi}_1: Z_1 \rightarrow \mathbb{C}^3/G_1$, where G_1 is of type $\frac{1}{4}(1, 0, 1)$;
- (ii) $\bar{\varphi}_2: Z_2 \rightarrow \mathbb{C}^3/G_2$, where G_2 is of type $\frac{1}{20}(4, 1, 5)$;
- (iii) $\bar{\varphi}_3: Z_3 \rightarrow \mathbb{C}^3/G_3$, where G_3 is of type $\frac{1}{5}(4, 0, 1)$.

As is shown in Figure A.1, note that for $k = 1, 3$, the morphism $\bar{\varphi}_k: Z_k \rightarrow \mathbb{C}^3/G_k$ is given by

$$G_k\text{-Hilb } \mathbb{C}^3 \rightarrow \mathbb{C}^3/G_k.$$

Thus, to show the existence of a G -brickset \mathfrak{S} for $Z \rightarrow X$, it only remains to show there exists a G_2 -brickset for $\bar{\varphi}_2: Z_2 \rightarrow \mathbb{C}^3/G_2$ by Theorem 3.12. Considering the star subdivision of $\text{Cone}(e_1, v_4, e_3)$ at v_8 , one can see that there exists a stability parameter $\theta^{(2)}$ such that Z_2 is isomorphic to the birational component of the moduli space of $\theta^{(2)}$ -stable G_2 -constellations in a similar way to the case in the main

theorem. Therefore we can conclude that there exists a G -brickset² \mathfrak{S} for $Z \rightarrow X$.

Finally, we discuss why we cannot see the existence of θ . First, a parameter ψ satisfying (4.9) can be found, eg. ψ can be defined to be

$$\psi(\rho_i) = \begin{cases} -1 & \text{if } 0 \leq i \leq 18, \\ 1 & \text{if } i = 19, \\ 1 & \text{if } 20 \leq i \leq 38. \end{cases}$$

On the other hand, the linear map

$$\phi_\star = ((\phi_1)_\star, (\phi_2)_\star, (\phi_3)_\star) : \Theta \rightarrow \Theta^{(1)} \oplus \Theta^{(2)} \oplus \Theta^{(3)}$$

is not surjective. Therefore, we cannot tell if there exists a solution for (3.17). However, this does not mean that there are no parameters θ for the G -brickset \mathfrak{S} . Using a computer, we might be able to find a parameter θ such that every $\Gamma \in \mathfrak{S}$ is θ -stable.

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²You can find the G -brickset \mathfrak{S} on my website:
<http://newton.kias.re.kr/~seungjo/CI1.html>

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